A GENERALIZATION OF THE FLEXIBLE LAW[1]

BY
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1. **Introduction.** Various identities have been substituted for the associative law. Many of these use the associator:

$$(x,y,z) = (xy)z - x(yz).$$

In this paper, we will call a ring R a (π,ϵ) ring if:

$$(x,x,x) = 0, \tag{1}$$

$$(x,y,z) = \epsilon(\pi(x),\ \pi(y),\ \pi(z)), \tag{2}$$

where for a in R, a/ϵ exists in R and π is in S_3, the symmetric group on three letters. It is easily shown that $\epsilon = \pm 1$ or $(x,y,z) = \epsilon(y,z,x)$, where $\epsilon^2 + \epsilon + 1 = 0$.

Throughout this paper an algebra $(A, +, x, F)$ is defined as a set A, a field F and two operations such that $(A, +, x)$ is a nonassociative ring, $(A, +, F)$ is a finite-dimensional vector space and such that $(\alpha x)(\beta y) = (\alpha\beta)(xy)$ if α,β are in F and x,y are in A. We shall, furthermore, find it convenient to define the following symbols:

$$\begin{aligned} x \cdot y &= xy + yx \qquad \text{(char. not prime to 2)}, \\ x \cdot y &= \frac{1}{2}(xy + yx) \qquad \text{(char. prime to 2)}. \end{aligned} \tag{3}$$

The ring formed by this multiplication is denoted R^+.

$$\{x * y * z\} = (x \cdot y) \cdot z - x \cdot (y \cdot z), \tag{4}$$

$$(x,y) = xy - yx. \tag{5}$$

The ring formed by this multiplication is denoted R^-.

$$\{x\theta y\theta z\} = ((x,y),z) - (x,(y,z)). \tag{6}$$

A semisimple algebra is one whose radical is zero where the radical is the maximal nil ideal. We further define a simple algebra as a semisimple algebra without proper ideals. An idempotent is an element e with $e^2 = e \neq 0$.

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[1] This paper presents the major results of the author's Ph.D. dissertation written under L. A. Kokoris at the Illinois Institute of Technology. In addition, many of the results are generalized.

In §3, necessary and sufficient conditions will be given for the following theorems to hold in an algebra.

THEOREM 1.1. *For every idempotent e,*

$$A = A_{11}(e) + A_{10}(e) + A_{01}(e) + A_{00}(e).$$

THEOREM 1.2. *The algebra A is central simple if and only if A is associative or nodal over some algebraic extension of F.*

THEOREM 1.3. *If A is semisimple, then A is a direct sum of simple algebras.*

Defining $x^\alpha = x^{\alpha-1}x$, we call a ring power-associative if $x^{\alpha+\beta} = x^\alpha x^\beta$ for all integers α, β. We will also need the following definitions.

DEFINITION 1.1. A ring is antiflexible if it satisfies $(x,y,z) = (z,y,x)$.

DEFINITION 1.2. A ring is nearly antiflexible if it satisfies

$$(x,x,y) = (y,x,x).$$

After some preliminary facts on antiflexible algebras and nearly antiflexible algebras in §4, the idempotent decomposition relative to a set of orthogonal idempotents is given in §5. In §6 it is shown that for char. $\neq 2$, a simple power-associative antiflexible algebra has an identity. A semisimple nearly antiflexible power-associative algebra is proved to be the vector space sum of an alternative ideal, degree one subspace and possibly a nil space. An alteration of the multiplication of nearly antiflexible algebras is given that preserves the nearly antiflexible identity and power-associativity. This alternation transforms a semisimple algebra into a direct sum of an alternative ideal, degree one ideals and possibly a nil ideal. A similar alteration is given for antiflexible algebras that gives slightly sharper results.

We will make use of the fact that a Lie ring is an anticommutative ring satisfying $(xy)z + (yz)x + (zx)y = 0$. A Jordan ring is a commutative ring satisfying $(x^2,y,x) = 0$. A ring R is Jordan admissible if R^+ is Jordan and Lie admissible if R^- is Lie.

2. (π,ϵ) **rings.** If a ring R is (π,ϵ_1) and (π,ϵ_2) then, upon subtraction,

$$0 = (x,y,z) - (x,y,z) = (\epsilon_1 - \epsilon_2)(\pi(x), \pi(y), \pi(z))$$

so either $\epsilon_1 = \epsilon_2$ or $(\pi(x), \pi(y), \pi(z)) = 0$. Thus we have the following result.

LEMMA 2.1. *If π is in S_3 and if R satisfies $(x,y,z) = \epsilon(\pi(x), \pi(y), \pi(z))$, then either ϵ is unique and nonzero or R is associative.*

LEMMA 2.2. *Let R satisfy $(x,y,z) = \epsilon(\pi(x), \pi(y), \pi(z))$. Then either R is associative or $\pi^n = I$ implies $\epsilon^n = 1$.*

Proof. Every element in S_3 is of order 1, 2, or 3 so

$$(x,y,z) = \epsilon(\pi(x), \pi(y), \pi(z)) = \epsilon^2(\pi^2(x), \pi^2(y), \pi^2(z)) = \epsilon^3(\pi^3(x), \pi^3(y), \pi^3(z))$$

implies, if $\pi^n = I$, $(x,y,z) = \epsilon^n(x,y,z)$. But this implies $(\epsilon^n - 1)(x,y,z) = 0$

from which the lemma follows.

F. Kosier, while not referring to them as (π, ϵ) algebras, studied $(\pi, \pm 1)$ algebras in [13]. The following lemma and its corollaries are readily justified.

LEMMA 2.3. *Let* π *be of order* 3 *and suppose* R *is a* (π, ϵ) *ring, then* R *is a* (π^2, ϵ^2) *ring.*

COROLLARY. *If* R *is* $((xyz), \epsilon)$ *then* R *is* $((xzy), \epsilon^2)$ *and if* R *is* $((xzy), \epsilon)$ *then* R *is* $((xyz), \epsilon^2)$.

COROLLARY. *A ring is* $((xyz), 1)$ *if and only if it is* $((xzy), 1)$. *If a ring is* $((xyz), \epsilon)$ *with* $\epsilon \neq 1$, *since* $\epsilon^3 = 1$, *if the ring is not associative then* $\epsilon^2 + \epsilon + 1 = 0$. *We have thus proved the main result of this section.*

THEOREM 2.1. *In a* (π, ϵ) *ring, either* $\epsilon = \pm 1$ *or* $\epsilon^2 + \epsilon + 1 = 0$ *and* $(x,y,z) = \epsilon(y,z,x)$.

In any ring, $(x,y,z) + (y,z,x) + (z,x,y) = (xy,z) + (yz,x) + (zx,y)$; so if $(x,y,z) + (y,z,x) + (z,x,y) = 0$ then $(xy,z) + (yz,x) + (zx,y) = 0 = (zy,x) + (yx,z) + (xz,y)$. Subtraction gives $((x,y),z) + ((y,z),x) + ((z,x),y) = 0$ so A^- is Lie. If A^- is Lie then $(x,y,z) + (y,z,x) + (z,x,y) = (z,y,x) + (x,z,y) + (y,x,z)$. If $(x,x,x) = 0$, then for char. prime to 2 we get $(x,y,z) + (y,z,x) + (z,x,y) = 0$. We now summarize these results.

LEMMA 2.4. (1) *If* $(x,y,z) + (y,z,x) + (z,x,y) = 0$ *then* A^- *is Lie.*

(2) *If* A^- *is Lie,* $(x,x,x) = 0$ *and char. prime to* 2 *then* $(x,y,z) + (y,z,x) + (z,x,y) = 0$.

3. **Necessary and sufficient conditions.** Define $R_{ij}(e) = \{x\colon ex = ix,\ xe = jx\}$. It is easily seen that if $R_{ij}(e)$ exists for some idempotent, then it is closed under addition. Also, if $R_{ij}(e)$ and $R_{km}(e)$ exist for distinct subscripts then only zero is common to both. If R is an algebra, then $R_{ij}(e)$ is a subspace.

For this section we will be interested in the following sets:

$\mathscr{A} = \{$Power-associative rings R such that if $e^2 = e$ then

(1) $R = R_{11}(e) + R_{10}(e) + R_{01}(e) + R_{00}(e)$.

(2) $R_{ij}(e)\, R_{km}(e) \subseteq R_{im}(e)$.

(3) $R_{ij}(e)\, R_{km}(e) = 0$ if $j \neq k$.

(4) $\mathscr{L}(e) = R_{10}(e)\, R_{01}(e) + R_{10}(e) + R_{01}(e) + R_{01}(e)\, R_{10}(e)$ is an associative ideal.$\}$

$\mathscr{B}^* = \{$Power-associative algebras A such that

(1) For every idempotent e, $A = A_{11}(e) + A_{10}(e) + A_{01}(e) + A_{00}(e)$.

(2) A is central simple if and only if A is associative or nodal over some algebraic extension of F.

(3) If A is semisimple then A is a direct sum of simple algebras which, if central simple, are in $\mathscr{B}^*$.$\}$

$\mathscr{D} = \{$Power-associative rings R such that for $e^2 = e$

(1) $(e,x,y) = (x,y,e) = (x,e,y) = 0$ for every x and y.

(2) If $R = R_{11}(e) + R_{10}(e) + R_{01}(e) + R_{00}(e)$ then $\mathscr{E}(e)$ is an associative ideal.}

$\mathscr{E} =$ {Power-associative rings R such that

(1) There is a k such that $x^k = 0$ or there is a linear combination

$$\sum_{i=1}^{n} \alpha_i x^i \neq 0$$

such that $(\sum_{i=1}^{n}\alpha_i x^i, y, z) = (y, \sum_{i=1}^{n} \alpha_i x^i, z) = (y, x, \sum_{i=1}^{n} \alpha_i x^i) = 0$ for every x, y and z in R.

(2) If $e^2 = e$ and $R = R_{11}(e) + R_{10}(e) + R_{01}(e) + R_{00}(e)$ then $\mathscr{E}(e)$ is an associative ideal.}

$\mathscr{F}^* =$ {Power-associative antiflexible rings.}

$[\mathscr{F}]^* =$ {Power-associative antiflexible algebras such that for every x, y and z there is either a k with $x^k = 0$ or there is a nonzero linear combination $\sum_{i=1}^{n} \alpha_i x^i \neq 0$ with $(\sum_{i=1}^{n}\alpha_i x^i, y, z) = 0$.}

$\mathscr{G} =$ {Power-associative algebras.}

$$\mathscr{A}^* = \mathscr{A} \cap \mathscr{G}.$$

$$\mathscr{D}^* = \mathscr{D} \cap \mathscr{G}.$$

$$\mathscr{E}^* = \mathscr{E} \cap \mathscr{G}.$$

$\mathscr{S} =$ {Semisimple algebras over an algebraically closed field.}

$$\mathscr{A}_s = \mathscr{A}^* \cap \mathscr{S}.$$

$$\mathscr{B}_s = \mathscr{B}^* \cap \mathscr{S}.$$

$$\mathscr{D}_s = \mathscr{D}^* \cap \mathscr{S}.$$

$$\mathscr{E}_s = \mathscr{E}^* \cap \mathscr{S}.$$

$$\mathscr{F}_s = \mathscr{F}^* \cap \mathscr{S}.$$

$$[\mathscr{F}_s] = [\mathscr{F}]^* \cap \mathscr{S}.$$

THEOREM 3.1. *The following relations hold for the above sets*:

(1) $\mathscr{A} = \mathscr{D}$.

(2) $\mathscr{A}^* = \mathscr{D}^* = \mathscr{E}^* \subseteq \mathscr{B}^*$.

(3) $\mathscr{A}_s = \mathscr{B}_s = \mathscr{D}_s = \mathscr{E}_s$.

(4) $\mathscr{A}_s \cap \mathscr{F} = \mathscr{B}_s \cap \mathscr{F} = \mathscr{D}_s \cap \mathscr{F} = \mathscr{E}_s \cap \mathscr{F}$.

Proof. Suppose R is in $\mathscr{A}$, x in $R_{ij}(e)$ and y in $R_{km}(e)$ then $(e,x,y) = (ix)y - i(xy) = 0$, $(x,y,e) = (m-m)(xy) = 0$ and $(x,e,y) = (j-k)(xy) = 0$ for if $j \neq k$ then $xy = 0$ so R is in $\mathscr{D}$. If R is in $\mathscr{D}$ then in R, $(x,e,e) = (e,x,e) = (e,e,x) = 0$ so from the proof of the associative case,

$$R = R_{11}(e) + R_{10}(e) + R_{01}(e) + R_{00}(e).$$

Furthermore, if x is in $R_{ij}(e)$ and y is in $R_{km}(e)$ then $(e,x,y) = (x,y,e) = 0$ implies xy is in $R_{im}(e)$ and $(x,e,y) = 0$ implies, for $j \neq k$, $R_{ij}(e)\, R_{km}(e) = 0$. Hence R is in $\mathscr{A}$ so $\mathscr{A} = \mathscr{D}$. Now, if A is in $\mathscr{A}^*$, using the methods employed to prove Theorems 3, 4 and 5 of [10], then A is in $\mathscr{B}^*$. Consider an element A of $\mathscr{E}^*$ and let e be an idempotent. Since $e^n = e$ and

$$\sum_{i=1}^{n} \alpha_i e^i = \left(\sum_{i=1}^{n} \alpha_i\right) e$$

then $\beta(e,x,y) = \beta(x,e,y) = \beta(x,y,e) = 0$, where $\beta = \sum_{i=1}^{n} \alpha_i \neq 0$ so A is in $\mathscr{D}$. Conversely, take A in $\mathscr{D}^*$. For any x in A, either x is nil so $x^k = 0$ for some k or x is not nil. If x is not nil then in the subalgebra generated by x there is an idempotent $e = \sum_{i=1}^{n} \alpha_i x^i$ for this subalgebra is non-nil and associative. (See [1].) Hence,

$$\left(\sum_{i=1}^{n} \alpha_i x^i, y, z\right) = \left(y, \sum_{i=1}^{n} \alpha_i x^i, z\right) = \left(y, z, \sum_{i=1}^{n} \alpha_i x^i\right) = 0$$

so A is in $\mathscr{E}^*$. We have proved $\mathscr{A}^* \subseteq \mathscr{B}^*$, $\mathscr{E} \subseteq \mathscr{D}$ and $\mathscr{D}^* \subseteq \mathscr{E}$ so (2) is true. If an algebra is in $\mathscr{B}_s$ and is simple then it is associative or nodal and is therefore in $\mathscr{A}_s$. Any algebra in $\mathscr{B}_s$ that has $\mathscr{L}(e)$ as an associative ideal is in $\mathscr{A}_s$ so we need only prove $\mathscr{L}(e)$ is an associative ideal. Suppose A is in $\mathscr{B}_s$ and is semisimple but not simple, then

$$A = A_{11}(e_1) \oplus A_{11}(e_2) \oplus \cdots \oplus A_{11}(e_n),$$

where $\sum_1^n e_i$ is an identity and each $A_{11}(e_i)$ is simple. Since F is algebraically closed then $A_{11}(e_i)$ is central simple and is in $\mathscr{B}_s$. Thus either e_i is primitive or $A_{11}(e_i)$ is associative. If $A_{11}(e_i)$ is associative and if u is an idempotent in $A_{11}(e_i)$ then $A_{10}(u) + A_{01}(u) \subseteq A_{11}(e_i)$ so $\mathscr{L}(u)$ is in $A_{11}(e_i)$ and is therefore associative. It is not difficult to show $\mathscr{L}(u)$ is an ideal in $A_{11}(e_i)$ using the methods used in [10] to show $\mathfrak{B}$ is an ideal. Clearly then $\mathscr{L}(u)$ is an associative ideal of A. If e is an arbitrary idempotent then $e = u_1 + \cdots + u_n$, where u_i is in $A_{11}(e_i)$ and thus $\mathscr{L}(e) = \mathscr{L}(u_1) \oplus \mathscr{L}(u_2) \oplus \cdots \oplus \mathscr{L}(u_n)$. (This can be proved by induction.) Therefore, $\mathscr{L}(e)$ is an associative ideal of A and hence A is in $\mathscr{A}_s$. We have proved (3) and now (4) is obvious.

Using the relations among $A_{ij}(e)$ proved by Kosier for antiflexible rings we get, if $x = x_{11} + x_{10} + x_{01} + x_{00}$ and $y = y_{11} + y_{10} + y_{01} + y_{00}$, where x_{ij}, y_{ij} are in $A_{ij}(e)$, $(x,e,y) = (x_{11} + x_{01})y - x(y_{11} + y_{10}) = 0$. Since $(e,x,y) = (y,x,e)$ and since $\mathscr{L}(e)$ is an associative ideal, then A is in $\mathscr{D}$ if and only if $(e,x,y) = 0$. It is clear that $\mathscr{E}^* \cap \mathscr{F} \subseteq [\mathscr{F}]^*$. Assuming A is in $[\mathscr{F}]^*$ we get for e an idempotent, $(x,e,y) = 0$ and $\mathscr{L}(e)$ is an associative ideal for $A = A_{11}(e) + A_{10}(e) + A_{01}(e) + A_{00}(e)$. Also $e^n = e$ and $\sum_{i=1}^{n} \alpha_i e^i = (\sum_{i=1}^{n} \alpha_i)e$ so $\beta(e,x,y) = 0$ for $\beta \neq 0$ so $(e,x,y) = 0$ for all x and y. Thus

$$(x,y,e) = (e,y,x) = 0$$

so A is in $\mathscr{D}^* \cap \mathscr{F}$. By Theorem 3.1, $\mathscr{D}^* = \mathscr{E}^*$ so $\mathscr{E}^* \cap \mathscr{F} = [\mathscr{F}]^*$.

THEOREM 3.2. (1) *The set* $[\mathscr{F}]^* = \mathscr{E}^* \cap \mathscr{F}$.

(2) *The set* $[\mathscr{F}_s] = \mathscr{A}_s \cap \mathscr{F} = \mathscr{B}_s \cap \mathscr{F} = \mathscr{D}_s \cap \mathscr{F} = \mathscr{E}_s \cap \mathscr{F}$.

We have thus found equivalent ways of defining $\mathscr{B}_s$. The results, as we have seen, are sharper for antiflexible algebras.

By applying the proofs in [10] to the special case $(x,y,z) = \epsilon(y,z,x)$ with $\epsilon^2 + \epsilon + 1 = 0$ we can easily prove this next theorem. The corollary is then obvious.

THEOREM 3.3. *If an algebra satisfies* $(x,y,z) = \epsilon(y,z,x)$ *with* $\epsilon^2 + \epsilon + 1 = 0$ *then it is in* $\mathscr{A}$, *provided it is power-associative.*

COROLLARY. *Any semisimple power-associative algebra satisfying* $(x,y,z) = \epsilon(y,z,x)$ *with* $\epsilon^2 + 1 = 0$ *over an algebraically closed field is in* $\mathscr{B}_s$.

4. **Antiflexible and nearly antiflexible rings.** So far we have two reasons for studying antiflexible rings. First, they were among the special cases of the identity used by Kosier [12]. They are also among the residual cases of (π, ϵ) rings. By using the associators of A^+ and A^- we will be able to give a third reason. A similar treatment will provide some new identities for nearly antiflexible rings. The following is easily proved by direct computation.

LEMMA 4.1. *In any ring* $K\{x * y * z\} + \{x\theta y\theta z\} = 2[(x,y,z) - (z,y,x)]$, *where* $K = 4$ *if char. is prime to* 2 *and otherwise* $K = 1$.

The identity $K\{x * y * z\} + \{x\theta y\theta z\} = 0$ then holds in any associative ring. We also have the following theorems and their corollaries.

THEOREM 4.1. *For char. prime to* 2, *a ring satisfies the identity* $4\{x * y * z\} + \{x\theta y\theta z\} = 0$ *if and only if the ring is antiflexible.*

COROLLARY. *For char. prime to* 2, *any two of the following statements about a ring R imply the third*:

(a) *The ring* R^+ *is associative.*

(b) *The ring* R^- *is associative.*

(c) *The ring R is antiflexible.*

THEOREM 4.2. *For char. prime to* 2, *a ring satisfies the identity*

$$4\{x * x * y\} + \{x\theta x\theta y\} = 0$$

if and only if the ring is nearly antiflexible.

COROLLARY. *For char. prime to* 6, *any two of the following statements about a ring R imply the third*:

(a) *The ring* R^+ *is associative.*

(b) *The ring* R^- *is alternative.*
(c) *The ring* R *is nearly antiflexible.*

Power-associative nearly antiflexible rings, though unnamed, were studied in [9]. Such a ring, if power-associative, decomposes relative to any idempotent into the subspaces $R = R_{11}(e) + R_{10}(e) + R_{01}(e) + R_{00}(e)$. It is further shown that $R_{ii}(e) \cdot R_{ii}(e) \subseteq R_{ii}(e)$, $R_{ii}^2(e) \subseteq R_{ii}(e) + R_{jj}(e)$, $R_{ij}(e)R_{ji}(e) \subseteq R_{ii}(e)$, $R_{ij}^2(e) \subseteq R_{ji}(e)$, $y_{ij}^2 = 0$, $R_{ii}(e)R_{jj}(e) = 0$, $R_{ij}(e)R_{jj}(e) \subseteq R_{ij}(e)$, and $R_{jj}(e)R_{ij}(e) = R_{ij}(e)R_{ii}(e) = 0$, where $i = 0$ or 1 and $j = 1 - i$. All but $R_{ii}(e) \cdot R_{ii}(e) \subseteq R_{ii}(e)$ are proved in [9] and this is proved in [2]. In addition, it is proved that $\mathscr{L}(e) = A_{10}(e)A_{01}(e) + A_{10}(e) + A_{01}(e) + A_{01}(e)A_{10}(e)$ is an alternative ideal. A few additional statements are true for antiflexible rings. Kosier [12] proved that in a power-associative antiflexible ring, $R_{ij}^2(e) = 0$. An antiflexible ring that satisfies $(x,x,x) = 0$ is Jordan and Lie admissible. For char. $= 0$, such a ring is power-associative. The ideal $\mathscr{L}(e)$ is associative in an antiflexible power-associative ring.

For Lie admissibility, Kosier did not use $(x,x,x) = 0$ so, if R is antiflexible, we have $\{x\theta y\theta z\} = ((x,y)z) - (x,(y,z)) = ((x,y)z) + ((y,z),x) = -((z,x),y)$. Combining this with Theorem 4.1 we have for char. prime to 2, $4\{x * y * z\} = ((z,x),y)$. Conversely, since $\{x * x * x\} = 0$ (for R^+ is commutative) and R^+ is Lie admissible (for $(R^+)^-$ is a zero ring) then, by Lemma 2.4, we have $\{x * y * z\} + \{y * z * x\} + \{z * x * y\} = 0$ for char. prime to 2. If, in addition, $4\{x * y * z\} = ((z,x),y)$ then R^- is Lie and $\{x\theta y\theta z\} = -((z,x),y)$ so, for char. prime to 2, $4\{x * y * z\} + \{x\theta y\theta z\} = 0$ and therefore R is antiflexible. We have proved the lemma.

Lemma 4.2. *For char. prime to* 2, *a ring is antiflexible if and only if* $4\{x * y * z\} = ((z,x),y)$.

From Lemma 2.4, since R^- is Lie if R is antiflexible, if $(x,x,x) = 0$, then $(x,y,z) + (y,z,x) + (z,x,y) = 0$ for char. prime to 2 and the converse is obvious if char. is prime to 3.

We have shown in Theorem 4.1 and Lemma 4.2 that the following three are equivalent for char. prime to 2:
(a) A ring is antiflexible.
(b) A ring satisfies $4\{x * y * z\} = -\{x\theta y\theta z\}$.
(c) A ring satisfies $4\{x * y * z\} = ((z,x),y)$.
It is interesting that from (c) we have for a power-associative antiflexible ring with char. prime to 2, $\{x^k * y * x^m\} = 0$.

Lemma 4.3. *For char. prime to* 6 *the following are equivalent for a ring*:
(a) *The ring is antiflexible and satisfies* $(x,x,x) = 0$.
(b) *The ring satisfies* $(xy,z) = x(y,z) - (z,x)y - 2(x,z,y)$.
(c) *The ring satisfies* $(x,y,z) + (y,z,x) + (y,x,z) = 0$.
(d) *The ring satisfies* $2(x \cdot y)z - 2y(x \cdot z) = (x,yz)$.

Proof. By direct and obvious computation for char. prime to 2, (d) and (c) are equivalent. In any ring, $(x,y,z) - (x,z,y) + (z,x,y) = (xy,z) - x(y,z) + (z,x)y$. Assume (c) is satisfied and permute x, y and z to get $(z,x,y) + (x,y,z) + (x,z,y) = 0$. Subtraction yields (b). Also on subtracting (b) from $(xy,z) = x(y,z) - (z,x)y + (x,y,z) - (x,z,y) + (z,x,y)$ which is true in any ring, we get (c) so (c) and (b) are always equivalent. For char. prime to 2 in an antiflexible ring with $(x,x,x) = 0$, $(z,x,y) = (y,x,z)$ and $(x,y,z) + (y,z,x) + (z,x,y) = 0$ so (c) is satisfied. Now assume (d). Interchanging y and z gives $2(x \cdot z)y - 2z(x \cdot y) = (x,zy)$. Subtraction yields $4\{y * x * z\} = (x,(y,z)) = ((z,y),x)$ so the ring is antiflexible. Letting $x = y = z$ in (d), we get $2x^2x - 2xx^2 = (x,x^2)$ or $3(x,x,x) = 0$ which for char. prime to 3 gives $(x,x,x) = 0$ so for char. prime to 6, (d) implies (a).

5. **Decomposition relative to a set of idempotents.** Let A be a nearly antiflexible power-associative algebra. If A has an identity element, then let $A = B$. If A does not have an identity, then let B be the algebra formed from A by attaching an identity element (see [14]). In an obvious way A can be thought of as an ideal of B. We will call this ideal A. From the assumption of finite dimensionality, if there is an idempotent in A then there is a principal idempotent e of A. This principal idempotent can be written as the sum of pairwise orthogonal idempotents. If $A \neq B$, $1 - e$ is an idempotent. If $1 - e$ is not primitive, then $\dim B \geqq \dim A + 2$ and this is impossible. Therefore $1 = \sum_{i=1}^{n} e_i$, where $\{e_i\}_{i=1}^{n}$ are pairwise orthogonal idempotents and either 1 or $\sum_{i=1}^{n-1} e_i$ is principal in A. If A is power-associative, then B is power-associative. Also, B is (nearly) antiflexible if and only if A is (nearly) antiflexible.

Decomposing B relative to orthogonal idempotents e and f will prove the following lemma. The proof will be omitted as it is computational and depends only on the idempotent decomposition of B.

LEMMA 5.1. *In B, let $ef = 0 = fe$, where $e^2 = e$ and $f^2 = f$ then*

(1) $B_{01}(e+f) = B_{00}(e) \cap B_{01}(f) + B_{01}(e) \cap B_{00}(f)$.

(2) $B_{10}(e+f) = B_{10}(e) \cap B_{00}(f) + B_{00}(e) \cap B_{10}(f)$.

(3) $B_{11}(e+f) = B_{11}(e) \cap B_{00}(f) + B_{10}(e) \cap B_{01}(f) + B_{01}(e) \cap B_{10}(f) + B_{00}(e) \cap B_{11}(f)$ *with* $B_{11}(e) = B_{11}(e) \cap B_{00}(f)$ *and* $B_{11}(f) = B_{00}(e) \cap B_{11}(f)$.

(4) $B_{00}(e+f) = B_{00}(e) \cap B_{00}(f)$.

(5) *If $e+f$ is an identity for B, then* $B_{10}(e) = B_{01}(f)$, $B_{01}(e) = B_{10}(f)$.

DEFINITION 5.1. $\beta_{ii} = B_{11}(e_i)$, $\beta_{ij} = B_{10}(e_i) \cap B_{01}(e_j)$, $\mathscr{A}_{ij} = \beta_{ij} \cap A$.

LEMMA 5.2. *In B, regardless of the numbering of the e_i:*

(1) $B_{00}(\sum_{i=1}^{t} e_i) = \bigcap_{i=1}^{t} B_{00}(e_i)$.

(2) $B_{01}(\sum_{i=1}^{t} e_i) \subseteq \sum_{i=1}^{t} B_{01}(e_i)$.

(3) $B_{10}(\sum_{i=1}^{t} e_i) \subseteq \sum_{i=1}^{t} B_{10}(e_i)$.

(4) $B_{11}(\sum_{i=1}^{t} e_i) = \sum_{i=1}^{t} \sum_{j=1}^{t} \beta_{ij}$.

Proof. Proofs of (1), (2), (3) and part of (4) will be by induction on t. Each of the statements is true for $t = 1, 2$ by Lemma 5.1. Now assume (1) for $t = k$. From above

$$B_{00}\left(\sum_{i=1}^{k+1} e_i\right) = B_{00}\left(\sum_{i=1}^{k} e_i\right) \cap B_{00}(e_{k+1}) = \bigcap_{i=1}^{k+1} B_{00}(e_i).$$

Assuming (2) for $t = k$ results in $B_{01}(\sum_{i=1}^{k+1} e_i) \subseteq B_{01}(\sum_{i=1}^{k} e_i) + B_{01}(e_{k+1}) \subseteq \sum_{i=1}^{k+1} B_{01}(e_i)$. The proof of (3) is similar. For (4), we first need to prove that $[\sum_{i=1}^{k} B_{10}(e_i)] \cap B_{01}(e_{k+1}) \subseteq \sum_{i=1}^{k} [B_{10}(e_i) \cap B_{01}(e_{k+1})]$. If we let x be in $[\sum_{i=1}^{k} B_{10}(e_i)] \cap B_{01}(e_{k+1})$ then $x = \sum_{i=1}^{k} x_i$ with x_i in $B_{10}(e_i)$ and x in $B_{01}(e_{k+1})$. Relative to e_{k+1}, $x_i = (x_i)_{11} + (x_i)_{10} + (x_i)_{01} + (x_i)_{00}$ so $\sum_{i=1}^{k} (x_i)_{11} = \sum_{i=1}^{k} (x_i)_{10} = \sum_{i=1}^{k} (x_i)_{00} = 0$. Consequently, $x = \sum_{i=1}^{k} (x_i)_{01}$ and x is in $\sum_{i=1}^{k} [B_{10}(e_i) \cap B_{01}(e_{k+1})]$. Interchanging 0 and 1 gives the fact that $[\sum_{i=1}^{k} B_{01}(e)] \cap B_{10}(e_{k+1}) \subseteq \sum_{i=1}^{k} [B_{01}(e_i) \cap B_{10}(e_{k+1})]$. Suppose $B_{11}(\sum_{i=1}^{k} e_i) = \sum_{i=1}^{k} \sum_{j=1}^{k} \beta_{ij}$. From the above lemma,

$$B_{11}\left(\sum_{i=1}^{k+1} e_i\right) = B_{11}\left(\sum_{i=1}^{k} e_i\right) + \left[B_{10}\left(\sum_{i=1}^{k} e_i\right)\right] \cap B_{01}(e_{k+1})$$
$$+ \left[B_{01}\left(\sum_{i=1}^{k} e_i\right)\right] \cap B_{10}(e_{k+1}) + B_{11}(e_{k+1}).$$

From (2), (3) of this lemma and the formulas just proved we derive the fact that $B_{11}(\sum_{i=1}^{k+1} e_i) \subseteq \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} \beta_{ij}$. From the distributive law it is clear that $\beta_{ij} \cap \beta_{mp} = 0$ if $i \neq m$ or $j \neq p$ so we can clearly see that $\sum_{i=1}^{k+1} \sum_{j=1}^{k+1} \beta_{ij} \subseteq B_{11}(\sum_{i=1}^{k+1} e_i)$. This completes the proofs.

Now $A = B \cap A$ and thus $A = (\sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij}) \cap A = \sum_{i=1}^{n} \sum_{j=1}^{n} (\beta_{ij} \cap A) = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathscr{A}_{ij}$.

THEOREM 5.1. *A nearly antiflexible, power-associative algebra A is the sum of the distinct subsets $\mathscr{A}_{ij}$.*

We will now derive the multiplication of the subspaces β_{ij}. We know that if β_{ij} and β_{km} are distinct that they have a zero intersection. Let x be in β_{ii} and let y be in β_{jj}. If $i \neq j$, $B_{11}(e_j) \subseteq B_{00}(e_i)$ because $B = B_{11}(e_i + e_j) + B_{10}(e_i + e_j) + B_{01}(e_i + e_j) + B_{00}(e_i + e_j)$ so

$$B_{11}(e_j) \cap (B_{11}(e_i) + B_{10}(e_i) + B_{01}(e_i)) = 0.$$

Thus for $i \neq j$, xy is in $B_{11}(e_i)B_{11}(e_j) \subseteq B_{11}(e_i)B_{00}(e_i) = 0$. It is known that $B_{11}(e_i) \cdot B_{11}(e_i) \subseteq B_{11}(e_i)$, so when $i = j$, $x \cdot y$ is in β_{ii}. We know that $B_{11}(e_i) \subseteq B_{00}(e_k)$ for all $k \neq i$ so x, y are in $B_{00}(e_k)$ for $k \neq i$. From [9] quoted before, xy is in $B_{11}(e_k) + B_{00}(e_k)$ for all $k \neq i$, and xy is in $B_{11}(e_i) + B_{00}(e_i)$. Now $B_{11}(e_k) \cap B_{11}(e_i) = 0, B_{11}(e_k) \cap B_{00}(e_i) = B_{11}(e_k), B_{11}(e_i) \cap B_{00}(e_k) = B_{11}(e_i)$ and $\bigcap_{i=1}^{n} B_{00}(e_i) = 0$ so xy is in $\sum_{i=1}^{n} \beta_{ii}$.

The proof of $\beta_{ij}^2 \subseteq \beta_{ji}$ follows from the facts that $B_{10}^2(e_i) \subseteq B_{01}(e_i)$ and $B_{01}^2(e_j) \subseteq B_{10}(e_j)$. If $i \neq j$ or $j \neq k$ it is easy to prove that $\beta_{ij}\beta_{jk} \subseteq \beta_{ik}$. We will illustrate the case $i \neq j = k$. We have $\beta_{ij}\beta_{jk} = [B_{10}(e_i) \cap B_{01}(e_j)][B_{11}(e_j)] = [B_{10}(e_i) \cap B_{01}(e_j)][B_{11}(e_j) \cap B_{00}(e_i)] \subseteq B_{10}(e_i) \cap B_{01}(e_j) \subset \beta_{ij}$ since $B_{11}(e_j) \subset B_{00}(e_i)$ for $i \neq j$. When $j \neq k$ and either $i \neq k$ or $j \neq m$ then it is not difficult to show $\beta_{ij}\beta_{km} = 0$. Now $\mathscr{A}_{ij}\mathscr{A}_{km} \subseteq \beta_{ij}\beta_{km}$ so the same rules hold for A. Hence we have this fact.

THEOREM 5.2. *In a nearly antiflexible power-associative algebra A, the subspaces $\mathscr{A}_{ij}$ satisfy the following laws:*

(1) $\sum_{i=1}^{n} \mathscr{A}_{ii}$ *is a subalgebra.*
(2) $\mathscr{A}_{ii} \cdot \mathscr{A}_{ii} \subseteq \mathscr{A}_{ii}$.
(3) $\mathscr{A}_{ij}^2 \subseteq \mathscr{A}_{ji}$.
(4) $\mathscr{A}_{ij}\mathscr{A}_{jk} \subseteq \mathscr{A}_{ik}$ *if $i \neq j$ or $j \neq k$.*
(5) $\mathscr{A}_{ij}\mathscr{A}_{km} = 0$ *if $j \neq k$ unless $i = k$ and $j = m$.*

These laws are similar to the laws in an alternative ring except for the fact that in an alternative ring, $\mathscr{A}_{ii}^2 \subseteq \mathscr{A}_{ii}$. In an antiflexible algebra $A_{10}^2(e) = A_{01}^2(e) = 0$ so we can replace (3) and (5) by a single formula.

THEOREM 5.3. *In an antiflexible power-associative algebra A the subspaces $\mathscr{A}_{ij}$ satisfy the laws:*

(1) $\sum_{i=1}^{n} \mathscr{A}_{ii}$ *is a subalgebra.*
(2) $\mathscr{A}_{ii} \cdot \mathscr{A}_{ii} \subseteq \mathscr{A}_{ii}$.
(3) $\mathscr{A}_{ij}\mathscr{A}_{km} = 0$ *if $j \neq k$.*
(4) $\mathscr{A}_{ij}\mathscr{A}_{jm} \subseteq A_{im}$ *if $i \neq j$ or $j \neq m$.*

The only difference between these laws and the corresponding laws for associative algebras is that in an associative algebra $\mathscr{A}_{ii}\mathscr{A}_{ii} \subseteq \mathscr{A}_{ii}$. However, Kosier [12] demonstrated that even a simple antiflexible power-associative algebra need not satisfy this. In different notation, an example of such an algebra is generated by e_1, e_2, x_1, x_2, y_1 and y_2 where $x_i^2 = y_i^2 = 0$, $e_i^2 = e_i$, $x_1y_1 = e_2$, $x_2y_2 = e_1$, $y_ix_i = -x_iy_i$, $(\alpha_1e_1 + \beta_1x_1 + \gamma_1y_1)(\alpha_2e_2 + \beta_2x_2 + \gamma_2y_2) = 0$ and $(\alpha_2e_2 + \beta_2x_2 + \gamma_2y_2)(\alpha_1e_1 + \beta_1x_1 + \gamma_1y_1) = 0$, where α_i, β_i and γ_i are in F.

6. **Simple antiflexible algebras.**

By our definition a simple algebra is non-nil and so has an idempotent. Kosier has shown [12] that for a simple, antiflexible, power-associative and not-associative algebra, if $e^2 = e$ then $A_{10}(e) + A_{01}(e) = 0$. He does this by showing in any antiflexible algebra, $\mathscr{L}(e)$ used in §3 is an associative ideal. The following set will be of importance in the rest of this paper where A is as defined in the previous section. From now on assume char. $\neq 2$. Furthermore, we use the word algebra for an antiflexible power-associative algebra.

DEFINITION 6.1. The set $Z = \{x \text{ in } A: (x, y) = 0 \text{ for all } y \text{ in } A\}$.

In A, $(xy,z) + (yz,x) + (zx,y) = (x,y,z) + (y,z,x) + (z,x,y) = 0$ so if z, x are in Z then $zx = xz$ is in Z. Clearly Z is a subspace so Z is a commutative subalgebra. For char. $\neq 2$, from the proof of Lemma 4.2, $(xy,z) = x(y,z) - (z,x)y - 2(x,z,y)$ and hence if z is in Z then $(x,z,y) = 0$, where x and y are arbitrary. Therefore, Z is associative as well as commutative. If A is simple then there is an idempotent e and, if A is not associative, then $A = A_{11}(e) + A_{00}(e)$. Consequently e is in Z so Z is not zero in a simple not-associative algebra. In an associative simple algebra there is a 1 which is in Z.

We know that in any ring $(wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z$ and $(zy,x,w) - (z,yx,w) + (z,y,xw) = z(y,x,w) + (z,y,x)w$. Using $(a,b,c) = (c,b,a)$, we have $(xw,y,z) - (w,yx,z) + (w,x,zy) = (x,y,z)w + (y,x,w)z$. Therefore $((w,x),y,z) - (w, (x,y),z) + (w,x,(y,z)) = (w,(x,y,z)) + ((w,x,y),z)$. Suppose x is in Z. From above $(w,x,(y,z)) = (w,x,y) = 0$ and since $(w,x) = (x,y) = 0$ then $(w,(x,y,z)) = 0$ so (x,y,z) is in Z. Since $(z,y,x) = (x,y,z)$ we have this result.

LEMMA 6.1. *Let x be in Z, then $(y,x,z) = 0$ and $(x,y,z) = (z,y,x)$ is in Z.*

Now let x,y be in Z and let a,b be in A. We can easily see that $(x + ya)b = xb + (ya)b = xb + y(ab) + (y,a,b), b(x + ya) = bx + b(ya) = xb + b(ay) = xb + (ba)y - (b,a,y) = xb + y(ba) - (y,a,b)$, where (y,a,b) is in Z. Hence $Z + ZA$ is an ideal in A. Since $Z \neq 0$ in a simple algebra then we have this fact.

LEMMA 6.2. *For char. $\neq 2$, if A is a simple antiflexible power-associative algebra then $A = Z + ZA$.*

The following definitions will hold in this and the next three sections. Since $xy = \sum_{i,j=1}^{n} z_{ij}$ with z_{ij} in $\mathscr{A}_{ij}$, let $xy \cap \mathscr{A}_{ij} = z_{ij}$.

DEFINITION 6.2. Let x,y be in $\mathscr{A}_{ii}$ and define $(xy)_k$, $\langle xy \rangle$ by: $(xy)_k = xy \cap \mathscr{A}_{kk}$, $\langle xy \rangle = xy \cap (\sum_{j \neq i} \mathscr{A}_{jj}) = \sum_{j\neq i} (xy)_j$.

If A is simple and not associative then, using the notation of the previous section, $A = \sum_{i=1}^{n} \mathscr{A}_{ii}$ for, from [12], if A is simple and not associative then $A = A_{11}(e) + A_{00}(e)$ relative to any idempotent e. Suppose x,y are in $\mathscr{A}_{ii}$ and z is in $\mathscr{A}_{kk}$, where $k \neq i$. It is clear that $(x,y,z) = (xy)z - x(yz) = (xy)_k z$ and $(z,y,x) = -z(yx)_k$. It has been shown that $\mathscr{A}_{ii} \cdot \mathscr{A}_{ii} \subseteq \mathscr{A}_{ii}$ so $0 = 2(x \cdot y)_k = (xy)_k + (yx)_k$. Therefore $(xy)_k$ is in Z, where $k \neq i$ and $\langle xy \rangle$ is in Z. Thus $\mathscr{A}_{ii}^2 \cap \sum_{j\neq i} \mathscr{A}_{jj} \subseteq Z$. We use Z' to mean the set theoretic complement of Z in A.

LEMMA 6.3. *Assume char. $\neq 2$ and let A be a simple antiflexible power-associative algebra which is not associative. Then $\mathscr{A}_{ii} \cap Z \neq 0$ and $\mathscr{A}_{ii} \cap Z' \neq 0$ for all i.*

Proof. Suppose $\mathscr{A}_{ii} \cap Z' = 0$ for some i. Then $\mathscr{A}_{ii}^2 = \mathscr{A}_{ii} \cdot \mathscr{A}_{ii} \subseteq \mathscr{A}_{ii}$

so $\mathscr{A}_{ii}$ is a nonzero ideal. Hence, $A = \mathscr{A}_{ii}$ and is commutative and therefore associative for $A = Z$. This proves $\mathscr{A}_{ii} \cap Z' \neq 0$. Now let $\mathscr{A}_{ii} \cap Z = 0$ for some i. Let x, y be in $\mathscr{A}_{kk}$ for $k \neq i$ then since $(xy)_i$ is in Z we have $(xy)_i = 0$ for all x and y in $\sum_{j\neq i} \mathscr{A}_{jj}$. Therefore $\sum_{j\neq i} \mathscr{A}_{jj}$ is an ideal in A and since A is simple we conclude $\sum_{j\neq i} \mathscr{A}_{jj} = 0$. Hence $A = \mathscr{A}_{ii}$ and $Z \neq 0$ so $\mathscr{A}_{ii} \cap Z \neq 0$.

The following lemma is easily verified by induction on m and Lemma 6.1.

LEMMA 6.4. *Let $\{a_i\}_{i=1}^m \subseteq Z$ then, for any x in A,*

$$\left(\prod_{i=1}^m a_i\right)x = a_1(a_2(a_3 \cdots (a_m x) \cdots)).$$

LEMMA 6.5. *Let the algebra A be simple, power-associative, antiflexible and not associative with char. $\neq 2$. For any i, there is an $x \neq 0$ in $\mathscr{A}_{ii}$ with $x = ax$, where a is in $Z \cap \mathscr{A}_{ii}$.*

Proof. Part I. We will first prove that for any x not in Z but in $\mathscr{A}_{ii}$ there is a set of elements $a_1, a_2, \cdots, a_m, b, y_1, \cdots, y_m$ with $\{a_i\}_{i=1}^m \cup \{b\} \subseteq Z \cap \mathscr{A}_{ii}$ and $\{y_1, \cdots, y_m\} \subseteq Z' \cap \mathscr{A}_{ii}$ such that $x = \sum_{i=1}^m a_i y_i + b$. From Lemma 6.2, $x = \sum_{i=1}^m a_i' y_i' + b'$, where $\{a_i'\}_{i=1}^m \cup \{b'\} \subseteq Z$ and $\{y_i'\} \subseteq Z'$ for if some y_i' is in Z then $a_i' y_i'$ is in Z and this can be included in b'. Let $a_i' = a_i + a_i''$, $b' = b + b''$, $y_i' = y_i + y_i''$, where a_i, b, y_i are in $\mathscr{A}_{ii}$ and a_i'', b'', y_i'' are in $\sum_{j\neq i} \mathscr{A}_{jj}$. If c' is in Z then $c' = c + c''$, where c is in $\mathscr{A}_{ii}$ and c'' is in $\sum_{j\neq i} \mathscr{A}_{jj}$. Let q be an arbitrary element of A. We have $q = r + s$ with r in $\mathscr{A}_{ii}$ and $\{s\} \subseteq \sum_{j\neq i} \mathscr{A}_{jj}$. Since c' is in Z then $c'q = qc'$, $c'r = rc'$ and $c's = sc'$. Also we know that $rc'' = cs = c''r = sc = 0$. Therefore, $cr = c'r = rc' = rc$ so that $cq = cr = rc = qc$ and c is in Z. Similarly c'' is in Z. Hence $\{a_i\} \cup \{b\} \subseteq Z \cap \mathscr{A}_{ii}$ and $\{a_i''\} \cup \{b''\} \subseteq Z \cap \sum_{j\neq i} \mathscr{A}_{jj}$. Clearly

$$\sum_{i=1}^m a_i' y_i' + b' = \sum_{i=1}^m a_i y_i + b + \sum_{i=1}^m a_i y_i'' + \sum_{i=1}^m a_i'' y_i + \sum_{i=1}^m a_i'' y_i'' + b''.$$

Since $\mathscr{A}_{ii}(\sum_{j\neq i} \mathscr{A}_{jj}) = 0$ then $a_i y_i'' = a_i'' y_i = 0$. Also, since a_i'' is in Z, then $a_i'' y_i'' = a_i'' \cdot y_i''$ is in $\sum_{j\neq i} \mathscr{A}_{jj}$ and b'' is in $\sum_{j\neq i} \mathscr{A}_{jj}$ so that $x = \sum_{i=1}^m a_i y_i + b$ and we have proved Part I.

Part II. We know that $\mathscr{A}_{ii} \cap Z' \neq 0$ so take x in $\mathscr{A}_{ii} \cap Z'$, where $x \neq 0$. $x = y_0 = \sum_{i=1}^{n_1} a_{1i} y_{1i} + d_1$, where $\{a_{1i}\}, d$ are in Z and $\{y_{1i}\}$ is not in Z and $\{a_{1i}\}, \{y_{1i}\}$ and d_1 are in $\mathscr{A}_{ii}$. Since x is not in Z then there is some y_{1,m_1} with $(a_{1,m_1})(y_{1,m_1})$ not in Z. Define $y_{1,m_1} = y_1$ and $a_{1,m_1} = a_1$. Since $a_1 y_1$ is not in Z then y_1 is not in Z for Z is a subalgebra. Suppose $\{a_k\}$, $\{y_k\}$ are defined and in $\mathscr{A}_{ii}$ for $k < j$ with the restrictions that $a_1(a_2 \cdots (a_k y_k) \cdots)$ is not in Z for all $k < j$. Since $a_1(a_2 \cdots (a_k y_k) \cdots) = (\prod_{i=1}^k a_i) y$, (from Lemma 6.4) is not in Z then y_k is not in Z for all $k < j$. Thus from Part I, $y_{j-1} = \sum_{i=1}^{n_j} a_{ji} y_{ji} + d_j$. There is a y_{j,m_j} such that $a_1(a_2 \cdots (a_{j,m_j} y_{j,m_j}) \cdots)$ is not in Z since

$$a_1(a_2 \cdots (a_{j-1} y_{j-1}) \cdots)$$

is not in Z. Define $y_{j,m_j} = y_j$ and $a_{j,m_j} = a_j$. Thus we get a set of elements $\{y_i\}$ not in Z and another set of elements $\{a_i\}$ in Z such that for every $j, a_1(a_2 \cdots (a_j y_j) \cdots)$ is not in Z. Define $r_j = a_1 a_2 \cdots a_j$ and

$$m_{jk} = a_{j+1} a_{j+2} \cdots a_k$$

so that $r_j m_{jk} = r_k$. From the finite demensionality of the subspace there is an s such that $\{r_j y_s\}_{j=1}^s$ are linearly dependent. From Lemma 6.4,

$$r_s y_s = a_1(a_2(a_3 \cdots (a_s y_s) \cdots))$$

which is not in Z by the construction of a_i and y_i. Since $r_s y_s$ is not in Z and r_k, m_{jk} are in Z for all j and k then $r_j y_s$ is not in Z for all j for, if $r_j y_s$ is in Z, then $r_s y_s = (m_{js} r_j) y_s = m_{js}(r_j y_s)$ is in Z. Now, let $z_i = r_i y_s$. Since $\{z_i\}$ is a set of linearly dependent vectors then $\sum_{i=1}^s \alpha_i z_i = 0$, where some α_i are not 0 and no z_i is zero for no z_i is in Z. Letting $t = \min\{i: \alpha_i \neq 0\}$ will give $z_t = -\sum_{i=t+1}^s (\alpha_i/\alpha_t) z_i = \sum_{i=t+1}^s \beta_i z_i$, where $\beta_i = -\alpha_i/\alpha_t$. We can clearly see that $z_i = r_i y_s = (m_{ji} r_j) y_s = m_{ji}(r_j y_s) = m_{ji} z_j$, where $j < i$ so for $i > t$ we have $z_i = m_{ti} z_t$. Therefore, $z_t = \sum_{i=t+1}^s \beta_i z_i = \sum_{i=t+1}^s (\beta_i m_{ti}) z_t = a z_t$, where

$$a = \sum_{i=t+1}^{s} \beta_i m_{ti}.$$

Since m_{ti} is in Z for all i then a is in Z and $x = z_t = ax$, where $x \neq 0$ and is in $\mathscr{A}_{ii}$.

Suppose A is simple and does not have an identity. An associative algebra, if simple, has an identity; so A is not associative. Therefore $A = \sum_{i=1}^n \mathscr{A}_{ii}$. If no $\mathscr{A}_{ii}$ is nil then there is an idempotent e_i in each $\mathscr{A}_{ii}$ and $\sum_{i=1}^n e_i$ is an identity. We thus have $A = \sum_{i=1}^n \mathscr{A}_{ii}$ and $\mathscr{A}_{nn}$ nil. From Lemma 6.5, there are a in $\mathscr{A}_{nn} \cap Z$ and $x \neq 0$ in $\mathscr{A}_{nn}$ with $x = ax$. Using Lemma 6.4 we get $x = ax = a^2 x = \cdots = a^{m-1} x = a^m x$ for all m. But x is not zero and for some m, $a^m = 0$ which is a contradiction, so A must have an identity.

THEOREM 6.1. *For char.* $\neq 2$, *a simple power-associative antiflexible algebra has an identity.*

7. **Semisimple algebras.** In this section, we will continue to employ the notation of the section on idempotent decomposition. Recall from [9] that $\mathscr{L}(e)$ is an alternative ideal if A is nearly antiflexible and power-associative. From the theorem of Artin [14], the subalgebra of an alternative algebra generated by two elements is associative. If x is in $B_{10}(e)$ and y is in $B_{01}(e)$, we have $(yx)^{k+1} = y(xy)^k x$ and $(xy)^{k+1} = x(yx)^k y$. Therefore,

$$\mathscr{L}_{11}(e) = \mathscr{L}_{10}(e)\mathscr{L}_{01}(e) = B_{10}(e) B_{01}(e)$$

is nil if and only if $\mathscr{L}_{00}(e) = \mathscr{L}_{01}(e)\, \mathscr{L}_{10}(e) = B_{01}(e) B_{10}(e)$ is nil. We also know that in B, $y_{10}^2 = y_{01}^2 = 0$ if y_{ij} is in $B_{ij}(e)$.

LEMMA 7.1. *In any nearly antiflexible power-associative algebra, for any idempotent, $\mathscr{S}(e)$ is nil if and only if $\mathscr{S}_{11}(e)$ is nil.*

Proof. It is obvious that $\mathscr{S}_{11}(e)$ is nil if $\mathscr{S}(e)$ is nil. Suppose $\mathscr{S}(e)$ is not nil but $\mathscr{S}_{11}(e)$ is nil. From above, $\mathscr{S}_{00}(e)$ is nil. Let N be the radical of $\mathscr{S}(e)$ and let $\mathscr{D} = \mathscr{S}(e)/N$ with $\mathscr{D}_{ij} = \mathscr{S}_{ij}(e)/N$. Now $\mathscr{D}$ is semisimple and so has an identity $v = v_{11} + v_{10} + v_{01} + v_{00}$, where v_{ij} is in $\mathscr{D}_{ij}$. The $\mathscr{D}_{ij}$ multiply as the $\mathscr{S}_{ij}(e)$ so $v_{11} + v_{10} + v_{01} + v_{00} = v = v^2 = v_{11}^2 + v_{10}v_{01} + v_{11}v_{10} + v_{10}v_{00} + v_{01}v_{11} + v_{00}v_{01} + v_{01}v_{10} + v_{00}^2$. Consequently, $v_{11} = v_{11}^2 + v_{10}v_{01}$, $v_{10} = v_{11}v_{10} + v_{10}v_{00}$, $v_{01} = v_{01}v_{11} + v_{00}v_{01}$ and $v_{00} = v_{01}v_{10} + v_{00}^2$. Since v is an identity for $\mathscr{D}$ then $v_{10} = v_{10}v = v_{10}v_{01} + v_{10}v_{00}$ so $v_{10} = v_{10}v_{00}$ and $v_{10}v_{01} = 0$; $v_{01} = v_{01}v = v_{01}v_{10} + v_{01}v_{11}$ so $v_{01} = v_{01}v_{11}$ and $v_{01}v_{10} = 0$. From above then $v_{11} = v_{11}^2 + v_{10}v_{01} = v_{11}^2$ and $v_{00} = v_{00}^2 + v_{01}v_{10} = v_{00}^2$. Hence $v_{11} = v_{11}^k$, $v_{00} = v_{00}^k$ for all k and, since $\mathscr{S}_{11}(e)$, $\mathscr{S}_{00}(e)$ are nil, then $\mathscr{D}_{11}$, $\mathscr{D}_{00}$ are nil so $v_{11} = v_{00} = 0$. Thus $v_{01} = v_{01}v_{11} = 0$, and $v_{10} = v_{10}v_{00} = 0$ and $v = 0$. But this contradicts the semisimplicity of $\mathscr{D}$.

If $e^2 = e \neq 0$ is primitive and $f^2 = f \neq 0$ with $ef = fe = f$, then $(e-f)^2 = e - f$, $(e-f)f = 0 = f(e-f)$ so $e = (e-f) + f$, where $e - f$ and f are orthogonal and f is an idempotent. Since e is primitive then $(e-f)^2 = e - f$ cannot be an idempotent so $e - f = 0$ and $e = f$. We will also need the following fact. The algebra B defined in terms of A in the section on idempotent decomposition is semisimple if A is semisimple.

THEOREM 7.1. *Let A be a nearly antiflexible, power-associative semisimple algebra and let e be a primitive idempotent of B, the algebra defined in the section on idempotent decomposition. Then $\mathscr{S}(e) = 0$ or $\mathscr{S}_{11}(e) = B_{11}(e)$.*

Proof. Assume $\mathscr{S}(e) \neq 0$. We know that B is semisimple so $\mathscr{S}(e)$ is non-nil. By Lemma 7.1, $\mathscr{S}_{11}(e)$ is non-nil and, since it is alternative, there is an idempotent u in $\mathscr{S}_{11}(e)$ with $ue = eu = u$. Since e is primitive, then $e = u$. Now, if y is in $B_{11}(e)$, then $ye = y$ is in $\mathscr{S}(e)$ for $\mathscr{S}(e)$ is an ideal. Therefore, y is in $B_{11}(e) \cap \mathscr{S}(e) = \mathscr{S}_{11}(e)$. Clearly $\mathscr{S}_{11}(e) \subseteq B_{11}(e)$ and we have proved that $\mathscr{S}_{11}(e) = B_{11}(e)$.

We know that $\sum_{i=1}^n \mathscr{A}_{ii}$ is a subalgebra so if x, y are in $\mathscr{A}_{ii}$ then

$$xy = \sum_{j=1}^n (xy)_j,$$

where $(xy)_j$ is in $\mathscr{A}_{jj}$. Define Z and $\langle xy \rangle$ for nearly antiflexible algebras as they were defined in §6 for antiflexible algebras.

LEMMA 7.2. *Z is an associative commutative subalgebra if char. $\neq 2,3$.*

Proof. In any ring with $(x,x,x) = 0$ and char. $\neq 2$ we have $0 = (x,y,z) + (y,z,x) + (z,x,y) + (z,y,x) + (y,x,z) + (x,z,y) = 2(x \cdot y, z) + 2(y \cdot z, x) + 2(z \cdot x, y)$. Therefore, if x, y are in Z we obtain $(xy, z) = (x \cdot y, z) = 0$ and hence Z is a subalgebra. From $xy = yx$ in Z we get the fact that Z is flexible and hence alternative for $(x,x,y)+(y,x,x)=0$. However, Z is commutative so $0=(xy,z)+(yz,x)+(zx,y)=(x,y,z)+(y,z,x)+(z,x,y)=3(x,y,z)$ and Z is associative.

Linearization of $(x,x,y)=(y,x,x)$ gives $(x,z,y)+(z,x,y)=(y,x,z)+(y,z,x)$. If x,y are in $\mathscr{A}_{ii}$ and z is in $\mathscr{A}_{jj}$ for $j\neq i$, we have $xz=zy=yz=zx=0$ and therefore $(z,x,y)=(y,x,z)$. It is clear that if $k\neq j$, $z(xy)_k=(yx)_k z=0$ so $z(xy)=z(xy)_j$ and $(yx)z=(yx)_j z$. Since $(z,x,y)=(y,x,z)$, we obtain the result $-z(xy)_j=(yx)_j z$ and from $\mathscr{A}_{ii}\cdot\mathscr{A}_{ii}\subseteq\mathscr{A}_{ii}$ it follows that $(xy)_j=-(yx)_j$. Hence, $z(xy)_j=(xy)_j z$. If z is in $\mathscr{A}_{kk}$ for $k\neq j, z(xy)_j=(xy)_j z=0$. We will use this to prove this fact.

Theorem 7.2. *In a semisimple power-associative nearly antiflexible algebra,* $\mathscr{A}_{ii}^2\cap(\sum_{j\neq i \text{ or } k\neq i}\mathscr{A}_{jk})\subseteq Z$.

Proof. If A is semisimple, let all of the e_i be primitive with $\mathscr{A}_{nn}$ possibly nil. From previous comments, in B, $\{e_i\}_{i=1}^n$ is a set of orthogonal primitive idempotents. Now take x,y in β_{ii}. If $\mathscr{L}_{11}(e_i)=\beta_{ii}$ then xy is in β_{ii} and $\langle xy\rangle=0$. If $\mathscr{L}_{11}(e_i)\neq\beta_{ii}$ then $\beta_{ij}=\beta_{ji}=0$ for all j. If z is in β_{km} with $k\neq i\neq m$ we have $xz=zx=zy=yz=0$ so as before $\langle xy\rangle z=z\langle xy\rangle$ and $\langle xy\rangle$ is in Z. Now Z in B intersected with A yields Z in A. Consequently, the proof follows.

Theorem 7.3. *Let A be a semisimple nearly antiflexible power-associative algebra. Then $A=C+D$ where C is an alternative ideal and $D=\sum_{i=m+1}^n\mathscr{A}_{ii}$ for some renumbering of the e_i and for some m (possibly zero) or $D=0$.*

Proof. Assume that A is not alternative. If this is so, then, by Theorem 7.1, $\mathscr{L}(e_i)=0$ for some i. In this case $D\neq 0$. Renumber the e_i in B so that $\mathscr{L}(e_i)\neq 0$ for all $i\leqq m$ and, if $\mathscr{A}_{nn}$ is nil, continue to denote e_n by e_n. Define $C=\sum_{i=1}^m\mathscr{L}(e_i)$. C is an alternative ideal since $\mathscr{L}(e_i)$ is an alternative ideal for each i and for $i>m$, $\mathscr{L}(e_i)=0$ by Theorem 7.1.

If A is antiflexible, $\mathscr{L}(e)$ is associative so the C of Theorem 7.3 is associative. Hence, we have this slightly sharper result for antiflexible algebras.

Theorem 7.4. *Let A be a semisimple antiflexible power-associative algebra. Then $A=C+D$, where C is an associative ideal and $D=\sum_{i=m+1}^n\mathscr{A}_{ii}$ for some renumbering of the e_i and for some m (possibly zero) or $D=0$.*

8. **An altered multiplication for antiflexible algebras.** It is frequently desirable to know, not only how far the theory of a given generalization may depart from the usual theorems in a branch of nonassociative algebra, but how a particular algebra can be changed to conform to the usual theorems. The theorems of this present section show how the multiplication of a given semisimple or simple antiflexible algebra can be altered to make this algebra a direct sum of an associative algebra, degree one algebras and (in the semisimple case) a nil algebra. A partial converse is given showing how, from a certain collection of algebras, semisimple and simple antiflexible algebras can be constructed.

DEFINITION 8.1. Let $(A, x, +, F)$ be an algebra with Z as previously defined. Define $(A^0, 0, +, F, \phi)$ by:

(1) $(A^0, +, F) \equiv (A, +, F)$.

(2) ϕ is a function from $A \times A$ into Z such that:

(a) $\phi[x;y] = -\phi[y;x]$ for all x, y in A, where $[x;y]$ is an element in $A \times A$.

(b) $\phi[(a,b);c)] = 0$ for all a, b, c in A.

(c) $\phi[x;y] = 0$ if x or y is in Z.

(3) $(A^0, 0)$ is defined by $x^0y = xy + \phi[x;y]$.

For simplicity we will denote $(A^0, 0, +, F, \phi)$ by (A^0, ϕ). It is clear that (A^0, ϕ) is completely determined by A and $\phi)$.

DEFINITION 8.2. Let ${}_0A = A$ and ${}_nA = ({}_{n-1}A^0, \phi_n)$, where $\{\phi_i\}$ is a collection of functions satisfying Definition 8.1.

It is obvious that given A there may be no ϕ but the zero function that satisfies the requirements for ϕ. If $\phi = \phi_n\phi_{n-1}\cdots\phi_2\phi_1$ then ${}_nA = (A^0, \phi)$ by induction on n. Also if $\phi' = -\phi$ then $A = ((A^0)^0, \phi')$, where $A^0 = (A^0, \phi)$.

From the definition of $A^0(\phi) = (A^0, \phi)$, $(A^0(\phi))^+ = A^+$ for $x^0y + y^0x = 2x \cdot y$. Therefore $\{x * y * z\}$ in $A^0(\phi)$ equals $\{x * y * z\}$ in A. Also, $(z^0x - x^0z)^0y - y^0(z^0x - x^0z) = (z,x)^0y - y^0(z,x) + 2\phi[z;x]^0y - 2y^0\phi[z;x] = ((z,x),y) + 2\phi[z;x]y - 2y\phi[z;x] = ((z,x),y)$ so the value of $((z,x),y)$ is unchanged. Hence the difference $4\{x * y * z\} - ((z,x),y)$ is unchanged and we have from Lemma 4.2 this lemma.

LEMMA 8.1. *For char. $\neq 2$ the algebra A is antiflexible if and only if $A^0(\phi)$ is antiflexible.*

THEOREM 8.1. *Let char. $\neq 2$ and assume F is algebraically closed. If A is semisimple, antiflexible and power-associative then there is a $\bar{f}$ satisfying* (2) *of Definition* 8.1 *such that $\bar{A} = A^0(\bar{f}) = C \oplus \mathscr{A}_{m+1,m+1} \oplus \cdots \oplus \mathscr{A}_{nn}$, where $\mathscr{A}_{ii}$ is of degree one or nil for $i > m$ and C is associative.*

Proof. Define $\bar{f}[x;y] = -\langle xy \rangle$ for all x and y in $\mathscr{A}_{ii}$ for each i. Since $\langle xy \rangle = -\langle yx \rangle$ then $\bar{f}[x;y] = -\bar{f}[y;x]$ and if x or y is in Z then $\langle xy \rangle = 0$ so $\bar{f}[x;y] = 0$. We have proved that $\langle xy \rangle$ is in Z. Now $\mathscr{A}_{ii} \cdot \mathscr{A}_{ii} \subseteq \mathscr{A}_{ii}$ so that $((z,x),y) = 4\{x * y * z\} \subseteq \mathscr{A}_{ii}$ for all z, x, y in $\mathscr{A}_{ii}$ and $\langle((z,x),y)\rangle = 2\langle(z,x)y\rangle$ so $\langle(z,x)y\rangle = 0$ and $\bar{f}[(z,x);y] = 0$. If x is in $\mathscr{A}_{ii}$ and y is in $\mathscr{A}_{jj}$ define $\bar{f}[x;y] = 0$ for $i \neq j$. If x or y is in C, define $\bar{f}[x;y] = 0$. Also let $\bar{f}[\alpha x + \beta y; z] = \alpha\bar{f}[x;z] + \beta\bar{f}[y;z]$ so that $\bar{f}$ is defined for all x and y in $\sum_{i=m+1}^{n} \mathscr{A}_{ii} + C$ (using the notation of Theorem 7.3). We have defined $\bar{f}$ on all of $A \times A$. Thus $\bar{f}$ satisfies the various requirements of Definition 8.1. In $\bar{A}$, $\mathscr{A}_{ii}^2 \subseteq \mathscr{A}_{ii}$ for all i so each $\mathscr{A}_{ii}$ is an ideal for $i > m$. Since F is algebraiclosed and the e_i are primitive then the $\mathscr{A}_{ii}$ for $i > m$ are of degree one unless A fails to have an identity in which case $\mathscr{A}_{nn}$ is nil. C is also an ideal in $\bar{A}$ since it is an ideal in A. Therefore, $\bar{A} = C \oplus \mathscr{A}_{m+1,m+1} \oplus \cdots \oplus \mathscr{A}_{nn}$.

COROLLARY. *If A is simple, antiflexible, not associative, power-associative over an algebraically closed field F and if char. $\neq 2$ then $\bar{A} = \mathscr{A}_{11} \oplus \cdots \oplus \mathscr{A}_{nn}$, where if $N(\mathscr{A}_{ii})$ is the radical of $\mathscr{A}_{ii}$ in $\bar{A}$ then $\mathscr{A}_{ii}/N(\mathscr{A}_{ii})$ is associative or nodal.*

Proof. In this case, m of Theorem 8.1 is zero for we know that

$$A = \sum_{i=1}^{n} \mathscr{A}_{ii}$$

and there is an identity so each $\mathscr{A}_{ii}$ is of degree one. Thus $\mathscr{A}_{ii}/N(\mathscr{A}_{ii})$ is associative or nodal.

We now let $\{A_i\}_{i=1}^{k}$ be a collection of simple nodal, antiflexible, power-associative algebras over the same field F, $C = \sum_{i=k+1}^{N} C_i$ be an associative semisimple algebra that is a direct sum of simple algebras C_i over F. We shall write D for the direct sum of the A_i and N_i for the set of nil elements in A_i. We know A_i^+ is Jordan so if $xy = \{xy\}e_i + (xy)$, where (xy) is in N and $\{xy\}$ is in F, then $\{xy\} = -\{yx\}$. We used e_i for the identity of A_i. For all i, e_i is in Z in D and $((z,x),y) = 4\{x * y * z\}$ is in N_i if z, x, y are in A_i. Also, if x or y is in Z in A_i then $xy = x \cdot y$ is in N_i so $\{xy\} = 0$. Define $g[x;y] = \{xy\}[e_{(i+1)\mathrm{mod}\,k} - e_i]$ for all x, y in N_i and define g in this manner on all $N_i \times N_i$. Define $g[x;y]$ to be 0 if x is in A_i or $x = e_j$ and y is in A_j for $i \neq j$ and let $g[\alpha x + \beta y; z] = \alpha g[x;z] + \beta g[y;z]$. The function g is now defined on all of $D \times D$ into (Z in D) $= Z(D)$ and by our remarks above satisfies all of part (2) in Definition 8.1 so that $D^0(g)$ is antiflexible. For x in N_i, $x^k = x^{\cdot k}$ is in N_i so $g[x^k; x] = 0$ for all k. Thus powers in D and $D^0(g)$ coincide so $D^0(g)$ is power-associative. Let J be an ideal of $D^0(g)$ with $J \cap A_i \neq 0$ in $D^0(g)$. Let x be in $J \cap A_i$. Since A_i is simple there is a set $\{y_i\}_{i=1}^{m}$ and a z in N_i in D such that some product of x and the y_i denoted by y has the property that y is in N_i but yz is not in N_i. Since y is in N_i in D then the circle product of the y_i and x denoted by $\bar{y}$ is also in N_i and equals y. Hence y is in J and $(y^0 z)^0 e_{(i+1)(\mathrm{mod}\,k)} = (\{yz\}e_{(i+1)(\mathrm{mod}\,k)} + w)^0 e_{(i+1)(\mathrm{mod}\,k)} = \{yz\}e_{(i+1)(\mathrm{mod}\,k)}$ and $\{yz\} \neq 0$. Consequently $A_{(i+1)(\mathrm{mod}\,k)}$ in $D^0(g)$ is in J. If A_m is in J then by the same reasoning $A_{(m+1)(\mathrm{mod}\,k)}$ is in J so $D^0(g) = J$ and $D^0(g)$ is simple.

LEMMA 8.2. *The algebra $D^0(g)$ is simple, antiflexible and power-associative for char. $\neq 2$.*

Define Q to be the zero algebra over F of order $2N$ with generators $\{x_{ij}\}$, $j = 1, 2$; $i = 1, \cdots, N$. Denote $D^0(g) + C + Q$ by A and denote the identity of C_i by e_i. Let $f[x_{i1}; x_{i2}] = e_i$, $f[x_{ij}; x_{km}] = 0$ if $i \neq k$ or $j = m$, $f[x;y] = 0$ if either x or y is in $D^0(g) + C$, $f[x_{i2}; x_{i1}] = -e_i$ and $f[\alpha x + \beta y; z] = \alpha f[x;z] + \beta f[y;z]$. It is not difficult to see that f satisfies part (2) of Definition 8.1 so that $A^0(f)$ is antiflexible. All powers are preserved which implies the fact that $A^0(f)$ is power-associative. If N is a nil ideal of A then N must be in Q for $D^0(g) + C$ is semisimple. But this is impossible for, if x_{ij} is in N, then $x_{ij} \circ x_{i,j\pm 1} = \pm e_i$ which is not nil. We have the following partial converse of Theorem 8.1.

LEMMA 8.3. *The algebra* $(D^0(g) + C + Q)^0(f)$ *is semisimple, antiflexible and power-associative.*

In addition, it is clear that, if A^+ is associative and char. $\neq 2$, then there is a ϕ satisfying Definition 8.1 with $A^+ = A^0(\phi)$ for, when A^+ is associative, A^- is associative and (x,y) is in Z. Hence $\phi[x;y] = -(x,y)/2$ satisfies Definition 8.1. Clearly then, $A = (A^+)^0(-\phi)$.

9. **An altered multiplication for nearly antiflexible algebras.** Some, but not all of the results can be extended to nearly antiflexible algebras. However, their extension requires a slightly different set of conditions on ϕ.

DEFINITION 9.1. Let $(A, x, +, F)$ be an algebra with Z as previously defined. Define $(A^0, 0, +, F, \phi)$ by:

(1) $(A^0, +, F) \equiv (A, +, F)$.

(2) ϕ is a function from $A \times A$ into Z such that:

(a) $\phi[x;y] = -\phi[y;x]$ for all x, y in A, where $[x;y]$ is an element in $A \times A$.

(b) $\phi[(a,b);b] = 0$ for all a, b in A.

(c) $\phi[x;y] = 0$ if x or y is in Z.

(3) $(A^0, 0)$ is defined by $x \circ y = xy + \phi[x;y]$.

We now restate the theorems that can be extended to nearly antiflexible algebras. There are three important facts that should be noted. First, the results of §6 are not known for nearly antiflexible algebras so the theorems that use those results must be changed or omitted. Secondly, except for obvious alterations in the proofs, they may be repeated verbatim. Hence, we omit these proofs. These alterations involve part (b) of part (2) which is the only difference between the definitions. Thirdly, neither definition will serve both purposes. The definition of the previous section is too restrictive for use in nearly antiflexible algebras. In antiflexible algebras $\langle (z,x)y \rangle = 0$ and this fact is needed in proving Theorem 8.1. However, in nearly antiflexible algebras we can prove $\langle (z,x)x \rangle = 0$ but are unable to show that

$$\langle (z,x)y \rangle = 0.$$

In addition to this, the above definition can produce an algebra that is not antiflexible from one that is.

If we let ${}_0A = A$ and ${}_nA = ({}_{n-1}A^0, \phi_n)$, then we can find a ϕ with

$${}_nA = (A^0, \phi).$$

Furthermore $\phi' = -\phi$ has the property that, if $A^0 = (A^0, \phi)$ then

$$A = ((A^0)^0, \phi').$$

We now list the results for nearly antiflexible algebras.

LEMMA 9.1. *For char. $\neq 2$ the algebra A is nearly antiflexible if and only if $A^0(\phi)$ is nearly antiflexible.*

THEOREM 9.1. *Let char. $\neq 2$ and assume F is algebraically closed. If A is semisimple, nearly antiflexible and power-associative, then there is a ϕ such that $A^0(\phi) = C \oplus \mathscr{A}_{m+1,m+1} \oplus \cdots \oplus \mathscr{A}_{nn}$, where $\mathscr{A}_{ii}$ is of degree one or nil for $i > m$ and C is alternative.*

DEFINITION 9.2. The ϕ of Theorem 9.1 will be denoted by $\bar{f}$ and $\bar{A} = A^0(\bar{f})$.

COROLLARY. *If A is simple, antiflexible, not associative, power-associative over an algebraically closed field F and if char. $\neq 2$ then $\bar{A} = \mathscr{A}_{11} \oplus \cdots \oplus \mathscr{A}_{nn}$, where if $N(\mathscr{A}_{ii})$ is the radical of $\mathscr{A}_{ii}$ in $\bar{A}$ then $\mathscr{A}_{ii}/N(\mathscr{A}_{ii})$ is associative or nodal with the possible exception that A_{nn} may be nil.*

Let $\{A_i\}_{i=1}^k$ be a collection of nearly antiflexible nodal power-associative algebras. Also, let $\{C_i\}_{i=k+1}^N$ be a collection of simple alternative algebras. Let e_i be the identity of A_i for $i \leqq k$ and let e_i be the identity of C_i for $i > k$. Define Q to be the zero algebra of order $2N$ generated by $\{x_i, y_i\}_{i=1}^N$. If we write $D = A_1 \oplus \cdots \oplus A_k \oplus C_{k+1} \oplus \cdots \oplus C_n \oplus Q$ and define $g[x_i; y_i] = e_i$, $-g[x;y] = 0$ if x or y is in D/Q, $g[x_i; y_i] = 0$, and $g[y_i, x_i] = -e_i$ then g can be extended bilinearly on all of $D \times D$. Thus define, g satisfies Definition 9.1 and $D^0(g)$ is semisimple, nearly antiflexible and power-associative.

So far, it is not known whether or not simple nearly antiflexible algebras must have identities. Not much is known about nodal algebras for any of the particular identities of this paper. It would be nice to know when circle multiplication destroys simplicity or semisimplicity. Finally, all examples of simple antiflexible algebras have A^+ associative but no proof has been found. If A is nodal and if Z contains no nil elements and if for some $n \geqq 4$, $\{(\cdots((x_1, x_2), x_3)\cdots, x_n)\} = 0$ then $\{(\cdots((x_1, x_2), x_3)\cdots, x_{n-1})\}$ is in $N \cap Z = 0$ so inductively $((x_1, x_2), x_3) = 0$ and A^- is associative.

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